Stories in mathematical physics and physical mathematics

Physics Concerto

Ainesh Sanyal

March 26, 2025

Introduction

- Friends from the mathematics department have always asked me to explain concepts from quantum mechanics in a mathematical way.

Introduction

- Friends from the mathematics department have always asked me to explain concepts from quantum mechanics in a mathematical way.
- I have always miserably failed to convey my point across. Perhaps it is just my incompetence, or perhaps there is something intrinsically confusing?

Introduction

- Friends from the mathematics department have always asked me to explain concepts from quantum mechanics in a mathematical way.
- I have always miserably failed to convey my point across. Perhaps it is just my incompetence, or perhaps there is something intrinsically confusing?
- Sometimes I like doing physics like mathematics and sometimes I like doing things the other way.

Introduction

- Friends from the mathematics department have always asked me to explain concepts from quantum mechanics in a mathematical way.
- I have always miserably failed to convey my point across. Perhaps it is just my incompetence, or perhaps there is something intrinsically confusing?
- Sometimes I like doing physics like mathematics and sometimes I like doing things the other way.
- I will narrate to you 2 stories. The first one about quantum mechanics being put on rigorous mathematical footing. This one will be a little technical.
- The second about supersymmetric field theories and physical mathematics.

But what is a quantum state?

- We want to seek a more general definition of a state in quantum mechanics.

But what is a quantum state?

- We want to seek a more general definition of a state in quantum mechanics.
- But what is wrong?

But what is a quantum state?

- We want to seek a more general definition of a state in quantum mechanics.
- But what is wrong?
- Griffiths: A state is equivalent to a wavefunction $\psi(x)$. My problem:

$$\psi(x) = \langle x|\psi\rangle$$

- The above equation does not make sense to me. $|x\rangle$ does not belong to any Hilbert space.

But what is a quantum state?

- We want to seek a more general definition of a state in quantum mechanics.
- But what is wrong?
- Griffiths: A state is equivalent to a wavefunction $\psi(x)$. My problem:

$$\psi(x) = \langle x|\psi\rangle$$

- The above equation does not make sense to me. $|x\rangle$ does not belong to any Hilbert space.
- Perhaps infinite dimensional spaces are too complicated. Let us stick to finite dimensions. Alright, Sakurai says that a state is an object:

$$|\psi\rangle \in \mathcal{H} \qquad \psi \sim e^{i\theta}\psi.$$

Contd.

- Alright, perhaps states are just elements of the projective space (the space of lines).

Contd.

- Alright, perhaps states are just elements of the projective space (the space of lines).
- No! These are just the *pure* states.

Contd.

- Alright, perhaps states are just elements of the projective space (the space of lines).
- No! These are just the *pure* states.
- Where are the *mixed* states/ density matrices?

$$\rho = \sum_i w_i |\psi_i\rangle \langle\psi_i| \text{ such that } w_i > 0 \text{ and } \sum_i w_i = 1$$

Contd.

- Alright, perhaps states are just elements of the projective space (the space of lines).
- No! These are just the *pure* states.
- Where are the *mixed* states/ density matrices?

$$\rho = \sum_i w_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \text{ such that } w_i > 0 \text{ and } \sum_i w_i = 1$$

- Conclusion: All this is non-rigorous. We need something more general.

Restart

- What do we want?

Restart

- What do we want?
- **Preparation procedures:** *Prepare* a certain *physical system* in a distinguished state.

Restart

- What do we want?
- **Preparation procedures:** *Prepare* a certain *physical system* in a distinguished state.
- **Registration procedures:** Measure a particular *observable* on this state.

Restart

- What do we want?
- **Preparation procedures:** *Prepare* a certain *physical system* in a distinguished state.
- **Registration procedures:** Measure a particular *observable* on this state.
- **Mathematical description:** Two sets S (*States*) and E (*Effects*) and a map:

$$S \times E \to [0, 1] \,,$$

such that $(\rho, A) \to \rho(A) \in [0, 1]$ denotes the probability.

Huh?

- What is the system? We will say that the system is characterized by an *operator algebra*.

Huh?

- What is the system? We will say that the system is characterized by an *operator algebra*.
- An operator algebra is a *closed* subspace of bounded operators of some Hilbert space $\mathcal{B}(\mathcal{H})$.

Huh?

- What is the system? We will say that the system is characterized by an *operator algebra*.
- An operator algebra is a *closed* subspace of bounded operators of some Hilbert space $\mathcal{B}(\mathcal{H})$.
- These are fancy words. For us, $\mathcal{H} = \mathbb{C}^d$. $\mathcal{B}(\mathcal{H})$ is just complex $d \times d$ matrices. $\mathcal{A}$ is some closed subalgebra of matrices.
- *States:*

$$S_{\mathcal{A}} = \{\rho \in \mathcal{A}^* \mid \rho(A) \geq 0 \,\forall A \in \mathcal{A} \text{ and } \rho(\mathbf{1}) = 1\}$$

Huh?

- What is the system? We will say that the system is characterized by an *operator algebra*.
- An operator algebra is a *closed* subspace of bounded operators of some Hilbert space $\mathcal{B}(\mathcal{H})$.
- These are fancy words. For us, $\mathcal{H} = \mathbb{C}^d$. $\mathcal{B}(\mathcal{H})$ is just complex $d \times d$ matrices. $\mathcal{A}$ is some closed subalgebra of matrices.
- *States:*

$$S_{\mathcal{A}} = \{\rho \in \mathcal{A}^* \mid \rho(A) \geq 0 \, \forall A \in \mathcal{A} \text{ and } \rho(\mathbf{1}) = 1\}$$

- *Effects:*

$$E_{\mathcal{A}} = \{A \in \mathcal{A} \mid A \geq 0 \text{ and } A \leq \mathbf{1}\}$$

Necessary math interlude

- Given the algebra $\mathcal{A}$ (or more generally, a vector space), one defines the dual space of $\mathcal{A}$ (the space of *linear functionals)*:

$$\mathcal{A}^* = \{\rho : \mathcal{A} \to \mathbb{C} \mid \rho \text{ is linear}\}$$

Necessary math interlude

- Given the algebra $\mathcal{A}$ (or more generally, a vector space), one defines the dual space of $\mathcal{A}$ (the space of *linear functionals)*:

$$\mathcal{A}^* = \{\rho : \mathcal{A} \to \mathbb{C} \ | \ \rho \text{ is linear}\}$$

- Any $\rho \in \mathcal{A}^*$ can be *represented* by $\tilde{\rho} \in \mathcal{A}$ by:

$$\rho(A) = \mathrm{tr}(\tilde{\rho} A) \ \ \forall A \in \mathcal{A}$$

Necessary math interlude

- Given the algebra $\mathcal{A}$ (or more generally, a vector space), one defines the dual space of $\mathcal{A}$ (the space of *linear functionals)*:

$$\mathcal{A}^* = \{\rho : \mathcal{A} \to \mathbb{C} \mid \rho \text{ is linear}\}$$

- Any $\rho \in \mathcal{A}^*$ can be *represented* by $\tilde{\rho} \in \mathcal{A}$ by:

$$\rho(A) = \mathrm{tr}(\tilde{\rho} A) \;\; \forall A \in \mathcal{A}$$

- The other way around:

$$\tilde{\rho}_{kj} = \rho(|j\rangle \langle k|)$$

Necessary math interlude

- Given the algebra $\mathcal{A}$ (or more generally, a vector space), one defines the dual space of $\mathcal{A}$ (the space of *linear functionals)*:

$$\mathcal{A}^* = \{\rho : \mathcal{A} \to \mathbb{C} \ | \ \rho \text{ is linear}\}$$

- Any $\rho \in \mathcal{A}^*$ can be *represented* by $\tilde{\rho} \in \mathcal{A}$ by:

$$\rho(A) = \mathrm{tr}(\tilde{\rho} A) \ \ \forall A \in \mathcal{A}$$

- The other way around:

$$\tilde{\rho}_{kj} = \rho(|j\rangle \langle k|)$$

- Only true for finite dimensional algebras.

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.
- Remember states were just density matrices in QM (positive and normalised *trace-class* operators).

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.
- Remember states were just density matrices in QM (positive and normalised *trace-class* operators).
- What does that have to do with our definition of states? Recall our definition of states:

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.
- Remember states were just density matrices in QM (positive and normalised *trace-class* operators).
- What does that have to do with our definition of states? Recall our definition of states:

$$S_{\mathcal{A}} = \{\rho \in \mathcal{A}^* \mid \rho(A) \geq 0\, \forall A \in \mathcal{A} \text{ and } \rho(\mathbf{1}) = 1\}$$

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.
- Remember states were just density matrices in QM (positive and normalised *trace-class* operators).
- What does that have to do with our definition of states? Recall our definition of states:

$$S_{\mathcal{A}} = \{\rho \in \mathcal{A}^* \mid \rho(A) \geq 0 \, \forall A \in \mathcal{A} \text{ and } \rho(\mathbf{1}) = 1\}$$

- Translate the conditions on ρ as conditions on $\tilde{\rho}$:

$$\rho \geq 0 \implies \tilde{\rho} \geq 0$$

Quantum mechanics

- The algebra $\mathcal{A}$ is just complex $d \times d$ matrices.
- Remember states were just density matrices in QM (positive and normalised *trace-class* operators).
- What does that have to do with our definition of states? Recall our definition of states:

$$S_{\mathcal{A}} = \{\rho \in \mathcal{A}^* \mid \rho(A) \geq 0 \, \forall A \in \mathcal{A} \text{ and } \rho(\mathbf{1}) = 1\}$$

- Translate the conditions on ρ as conditions on $\tilde{\rho}$:

$$\rho \geq 0 \implies \tilde{\rho} \geq 0$$

$$\rho(\mathbf{I}) = \mathrm{tr}(\tilde{\rho}) = 1.$$

- Hence, $\tilde{\rho}$ are the density matrices!

What about the pure states?

- S and E are convex (easy to see).

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

What about the pure states?

- S and E are convex (easy to see).
- A convex has *extremal points* (another theorem).

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

What about the pure states?

- S and E are convex (easy to see).
- A convex has *extremal points* (another theorem).
- The extremal points of S are *the pure states*.

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

What about the pure states?

- S and E are convex (easy to see).
- A convex has *extremal points* (another theorem).
- The extremal points of S are *the pure states*.
- No artificial normalisation condition required. It is all built in.

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

What about the pure states?

- S and E are convex (easy to see).
- A convex has *extremal points* (another theorem).
- The extremal points of S are *the pure states*.
- No artificial normalisation condition required. It is all built in.
- Now that, we have a rigorous definition of states, we can do a lot more! I refer you to an article[1] for more.

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

What about the pure states?

- S and E are convex (easy to see).
- A convex has *extremal points* (another theorem).
- The extremal points of S are *the pure states*.
- No artificial normalisation condition required. It is all built in.
- Now that, we have a rigorous definition of states, we can do a lot more! I refer you to an article[1] for more.
- This was a story where we did physics mathematically. What about the other way around?

[1] Michael Keyl. "Fundamentals of quantum information theory". In: *Physics Reports* 369.5 (Oct. 2002), pp. 431–548. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00266-1. URL: http://dx.doi.org/10.1016/S0370-1573(02)00266-1.

The beginning of physical mathematics

- In 1994, Seiberg and Witten wrote a legendary paper[2] on answering non-perturbative questions about field theories.

[2] N. Seiberg and Edward Witten. "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426 (1994). [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

The beginning of physical mathematics

- In 1994, Seiberg and Witten wrote a legendary paper[2] on answering non-perturbative questions about field theories.
- Perturbative questions are hard.

[2] N. Seiberg and Edward Witten. "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426 (1994). [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

The beginning of physical mathematics

- In 1994, Seiberg and Witten wrote a legendary paper[2] on answering non-perturbative questions about field theories.
- Perturbative questions are hard.
- Non-perturbative questions are even *harder*.

[2] N. Seiberg and Edward Witten. "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426 (1994). [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

The beginning of physical mathematics

- In 1994, Seiberg and Witten wrote a legendary paper[2] on answering non-perturbative questions about field theories.
- Perturbative questions are hard.
- Non-perturbative questions are even *harder*.
- Seiberg and Witten translated a question in *physics* to a question in *geometry*.

[2] N. Seiberg and Edward Witten. "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426 (1994). [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

Two slides on technical details

- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$

Two slides on technical details

- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$
- **Goal:** Determine $\mathcal{F}$ exactly (with non-perturbative contributions).

Two slides on technical details

- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$
- **Goal:** Determine $\mathcal{F}$ exactly (with non-perturbative contributions). **Ingredients:**
  1. Holomorphy of $\mathcal{F}$.

Two slides on technical details

- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$
- **Goal:** Determine $\mathcal{F}$ exactly (with non-perturbative contributions). **Ingredients:**
  1. Holomorphy of $\mathcal{F}$.
  2. Physics input - 1-loop contribution.

Two slides on technical details

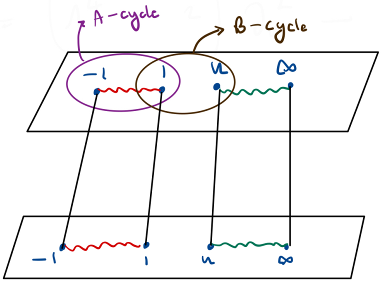
- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$
- **Goal:** Determine $\mathcal{F}$ exactly (with non-perturbative contributions). **Ingredients:**
  1. Holomorphy of $\mathcal{F}$.
  2. Physics input - 1-loop contribution.
  3. Electric-magnetic duality.

Two slides on technical details

- $SU(2)$ $\mathcal{N} = 2$ field theory in 4 dimensions. The *low-energy effective action* is completely described in terms of a holomorphic function $\mathcal{F}(a)$
- **Goal:** Determine $\mathcal{F}$ exactly (with non-perturbative contributions). **Ingredients:**
  1. Holomorphy of $\mathcal{F}$.
  2. Physics input - 1-loop contribution.
  3. Electric-magnetic duality.
  4. Monodromy.

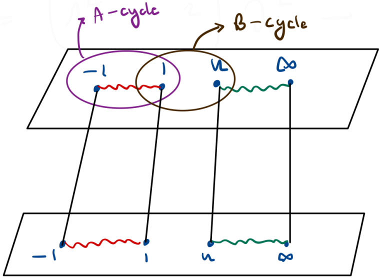
Contd.

- It turns out: Finding $\mathcal{F}$ is finding some property of the torus:



Contd.

- It turns out: Finding $\mathcal{F}$ is finding some property of the torus:



- I invite you to my talk at the Junior Geometry and Quantum Field Theory seminar next Monday if you are interested in this.

So what?

- It turns out this led to a revolution in studying supersymmetric field theories.

So what?

- It turns out this led to a revolution in studying supersymmetric field theories.
- Led to conjectures in mathematics (Donaldson-Thomas invariants, counting BPS states) One can go wild with these kinds of theories. There are even non-Lagrangian theories which we can study using these kinds of techniques.

So what?

- It turns out this led to a revolution in studying supersymmetric field theories.
- Led to conjectures in mathematics (Donaldson-Thomas invariants, counting BPS states) One can go wild with these kinds of theories. There are even non-Lagrangian theories which we can study using these kinds of techniques.
- Knot theory à la Witten.

So what?

- It turns out this led to a revolution in studying supersymmetric field theories.
- Led to conjectures in mathematics (Donaldson-Thomas invariants, counting BPS states) One can go wild with these kinds of theories. There are even non-Lagrangian theories which we can study using these kinds of techniques.
- Knot theory à la Witten.
- Hopefully, I have convinced you that both lines of thought are useful. Thank you for your patience!