An Introduction to Quantum Hall Physics

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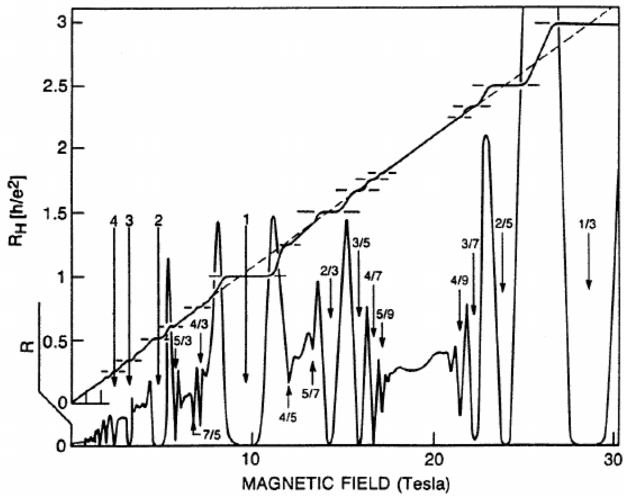
What is the Quantum Hall Effect?



[1]

[1] Girvin, Steven. (2004). Introduction to the Fractional Quantum Hall Effect. 10.1007/3-7643-7393-8$_4$.

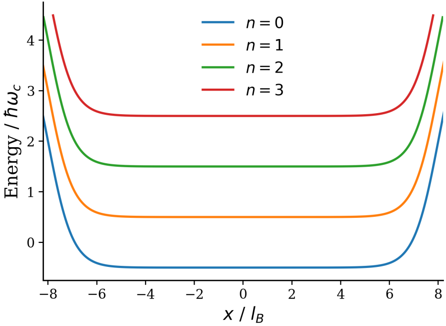
Thanks for listening!

Prereq: Landau Levels

Electron in a magnetic field:

$$\mathcal{H}_{Landau} = \frac{1}{2m_e}\left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right)^2 \tag{1}$$

Has energy levels: $E_n = (n + \frac{1}{2})\hbar\omega_c$ called the "Landau levels".



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[2] AR

Work in "symmetric gauge",i.e $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ and $z = x + iy$. Our Hamiltonian is then,

$$\mathcal{H}_{Landau} = \frac{\hbar^2}{m_e}\left[-2\left(\partial - \frac{\bar{z}}{4l_0^2}\right)\left(\bar{\partial} + \frac{z}{4l_0^2}\right) + \frac{1}{2l_0^2}\right] \tag{2}$$

with (non-normalized) eigenstates,

$$\psi_{n,m}(z,\bar{z}) = z^m L_n^m(z,\bar{z}) e^{-\frac{|z|^2}{4l_0^2}} \tag{3}$$

$E_n = (n + \frac{1}{2})\frac{h}{l_0}$, L_n^m are the Laguerre polynomials and $l_0 = \sqrt{\frac{\hbar}{eB}}$

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Lowest Landau Levels

The wavefunctions in the lowest Landau level are given by,

$$\psi_{n=0,m}(z,\bar{z}) = z^m e^{-\frac{|z|^2}{4l_0^2}} \tag{4}$$

In general these LLL wavefunctions can take the form,

$$\Psi_{LLL} = \sum_m \alpha_m \psi_{n=0,m}(z,\bar{z}) = f(z) e^{-\frac{|z|^2}{4l_0^2}} \tag{5}$$

Universal Properties

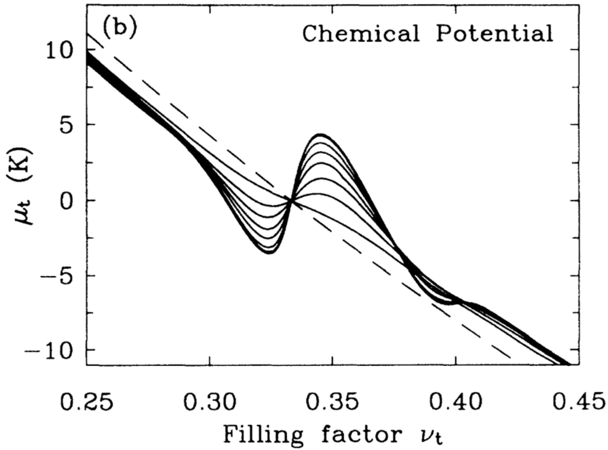
1. Incompressibility

Incompressibility : adding or removing electrons from the system costs a finite amount of energy

We define compressibility in the thermodynamic limit as,

$$\kappa^{-1} = n^2 \frac{\partial \mu}{\partial n} \tag{6}$$

- An incompressible state has $\kappa = 0$
- μ is discontinuous
- Gap in excitation spectrum , it resists changes in its density.



[3]

[3] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West. "Compressibility of the two-dimensional electron gas: Measurements of the zero-field exchange energy and fractional quantum Hall gap". In: *Phys. Rev. B* 50 (3 July 1994), pp. 1760–1778. DOI: 10.1103/PhysRevB.50.1760. URL: https://link.aps.org/doi/10.1103/PhysRevB.50.1760

- **Incompressibility implies edge currents**
- Suppose μ is in a charge gap, then some $\delta\mu$ cannot change the local current density in the bulk. **Can only affect the edge states.**
- Current is quantized. This is due to the following arguments: Using

  the Streda formula,

$$\sigma_{xy} = \frac{\partial n}{\partial B} \tag{7}$$

The Landau level degeneracy is given by, $N_{deg} = \frac{eBA}{h}$ and $n = \nu \frac{N_{deg}}{A}$. Substituting into the Streda formula gives,

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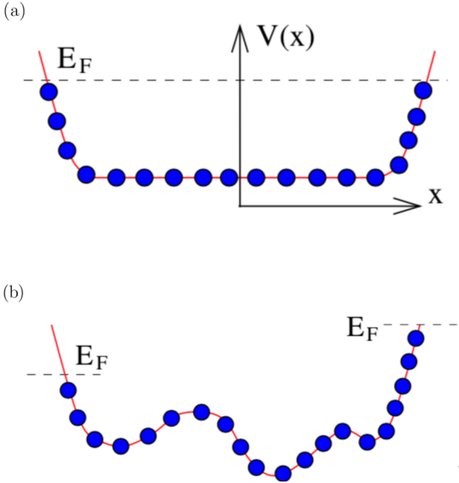


[4]

[4] âThe Integer Quantum Hall Effect.â(2018),Abouzaid, Aly et al.

2. Translational Symmetry

In a LL, the traditional translation operators do not commute with H. Instead we define "magnetic translation operators" that do,

$$T(x) = e^{ix^\mu D_\mu} = e^{ix^\mu (p_\mu + eA_\mu)} \tag{9}$$

Drawback: They don't commute with each other! We get the following,

$$T(a)T(b) = e^{i\phi(a,b)}T(b)T(a) \tag{10}$$

where $\phi(a,b) = 2\pi \frac{BA}{\Phi_D}$

In terms of the guiding center coordinate the same algebra can be written as,

$$e^{ipX} e^{iqX} = e^{i(p+q)X} e^{il_B^2 \frac{p \wedge q}{2}} \tag{11}$$

Girvin,Macdonald and Platzman used this result to derive a relation for the single-particle densities,

$$\left[\rho(q), \rho(q')\right] = 2isin(l_B^2\, q \wedge q'/2)\rho(q + q') \tag{12}$$

Insert fancy words: 'area-preserving diffeomorphisms', 'W_∞ algebra', 'Chern-Simons'

3. Edge Modes

- Bulk $=$ gapped , $\delta\rho_{bulk}$ $=$ bad
- Edge $=$ not gapped, $\delta\rho_{edge}$ $=$ good

Theory of edge modes: Chiral Luttinger Liquid[5][10]

Incompressibility and "non-commutative" geometry implies *gapless edge modes*

The edge densities satisfy[5],

$$[\hat{\rho}_k, \hat{\rho}_l] = -\frac{k}{2\pi m}\delta_{k,l} \tag{13}$$

[5] Zyun F. Ezawa. *Quantum Hall Effects: Field Theoretical Approach and Related Topics*. 2nd. Singapore: World Scientific, 2008. ISBN: 9789812791398. DOI: 10.1142/6739

Some dude [Wen] came up with a theory for the edge:

$$\rho_{L,R}(x) = \partial_x \phi_{L,R}(x) \tag{14}$$

$$\left[\phi_{L,R}(x), \phi_{L,R}(x')\right] = -\frac{i}{4\pi m} sgn(x - x') \tag{15}$$

with the following action,

$$S = \pi\hbar m \int dtdx \;\; 2\partial_t\phi\partial_x\phi - v\partial_x\phi\partial_x\phi + [contact\ int.] \tag{16}$$

4. What is Topological Order?

- Doesn't break any symmetry
- Is not characterized by local order parameters
- Involves long-range quantum entanglement
- Is robust against any local perturbations

QH liquid is a topologically ordered phase of matter

Integer Case: $\nu \in \mathbb{Z}$

In Quantum Mechanics we have operators (and wavefunctions).

[6]

[6] Eddy

- Fermion wavefunctions are slater determinants.
- Integer = filled landau levels.

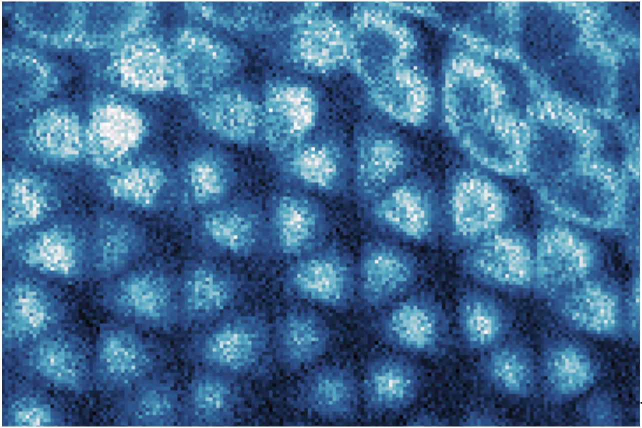
For $\nu = 1$,

$$\text{Wavefunction} = \prod_{i<j} (z_i - z_j) e^{-\sum_k |z_k|^2/4l_B^2} \quad (17)$$

What about the fractional case? $\nu = 1/m$

- Kinectic < Coloumb
- Strong repulsion between electrons

What happens?

[7]

[7] Wigner Crystal

FQH Ground State is a fluid too!

[8]

[8]Shocked

Why crystal bad?

- Want to preserve the topology
- Localizing electrons requires energy

Instead try this,

$$\Psi_{1/m} = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4l_0^2}} \tag{18}$$

This one comes with,

- Lower energy without breaking translation symm.
- Respects circular symmetry of magnetic field
- Electrons avoid each other strongly
- Incompressible


[9]

Robert Laughlin: Quantum hall enthusiast, winner of prizes

Charged Excitations

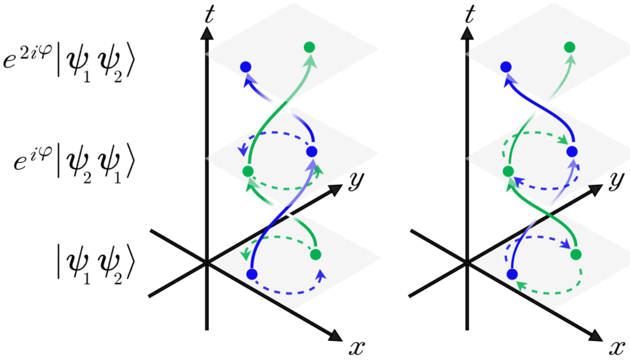
The charged excitations are vortices. Increasing the angular momentum of each electron by $+1$ we find,

$$\Psi_+[\eta] = \prod_{i=1}^{N} \prod_{j<k} (z_i - \eta)(z_k - z_j)^m e^{-\sum_p \frac{|z_p|^2}{4 l_0^2}} \tag{19}$$

- Charge depletion = 'quasihole'
- Vortex has fractional charge = e/m
- Fractional Statistics

Fractional Statistics

Charged excitations are NOT Bosons, NOT Fermions $\rightarrow$ 'Anyons'



[10]

[10] Anyons

Supplement

3. Laughlin's State

A breakthrough came from Laughlin in 83 [11], who suggested the following trial wavefunction for $\nu = \frac{1}{2s+1}$,

$$\Psi_{1/2s+1} = \prod_{i=1}^{N} \prod_{j<i} (z_i - z_j)^{2s+1} e^{-\sum_i \frac{|z_i|^2}{4l_0^2}} \tag{20}$$

[11] R. B. Laughlin. "Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations". In: *Phys. Rev. Lett.* 50 (18 May 1983), pp. 1395–1398. DOI: 10.1103/PhysRevLett.50.1395. URL: https://link.aps.org/doi/10.1103/PhysRevLett.50.1395

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What's so good about it?

- It lies in the LLL (analytic function $\times$ Gaussian)
- Anti-symmetric/Obeys the Pauli principle
- Particles avoid each other strongly (2s+1-fold zeros)
- Is exact for a class of short-ranged interactions (Haldane Pseudopotentials)
- Describes an incompressible fluid

4. Laughlin's Plasma Analogy

Let us evaluate $|\langle\Psi_{1/2s+1}|\Psi_{1/2s+1}\rangle|^2$ to get,

$$|\langle\Psi_{1/2s+1}|\Psi_{1/2s+1}\rangle|^2 = e^{-2\ln|\langle\Psi_{1/2s+1}|\Psi_{1/2s+1}\rangle|}$$
$$= \int \prod_p dz_p e^{-\frac{1}{q}\left(-2q^2\sum_{i<j}^{N}\ln|z_i - z_j| + \frac{q}{2}\sum_i^N \frac{|z_i|^2}{l_0^2}\right)} \quad (21)$$

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which corresponds to the classical Boltzmann distribution[3] of a 2D homogenous plasma of particles with charge q at temperature $T = \frac{q}{k_B}$.

- Plasma equilibrium when charge neutral $\rightarrow \frac{1}{2\pi l_0^2} = q\frac{N}{A}$ or $\nu = \frac{N}{N_0} = \frac{1}{q}$
- Adding an electron to the system requires overcoming the 2D Coloumb $\rightarrow$ **finite energy gap**
- $\delta\rho(r)$ **is logarithmically suppressed**

5. Excitations of FQH Ground State

- Neutral excitations (fluctuation of acoustic modes) that are "magneto-rotons" [Girvin, Macdonald, Platzmann][4] or particle-hole excitations
- Charged excitations that are quasiholes and quasiparticles ("anyons") [Laughlin]

Our focus today will be on the charged quasi-hole excitations

1 quasi-hole wavefunction

$$\Psi_{excited}[\eta] = \prod_{i=1}^{N} \prod_{j<k} (z_i - \eta)(z_k - z_j)^{2s+1} e^{-\sum_i \frac{|z_i|^2}{4l_0^2}} \tag{22}$$

6. Griffiths: Chapter 9

Statement of the adiabatic theorem: *"if a quantum system is subjected to a slowly changing Hamiltonian, then it will remain in its instantaneous eigenstate, provided there is a gap between the energy levels of the system."*

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For a system that varies with respect to some parameter $\lambda(t)$ the eigenstates $|j, \lambda\rangle$ vary as,

$$|(j, \lambda(0))(t)\rangle = e^{i\phi_j(t)}|j, \lambda(t/T)\rangle \tag{23}$$

where $\phi_j(t)$ is called the "Berry phase".

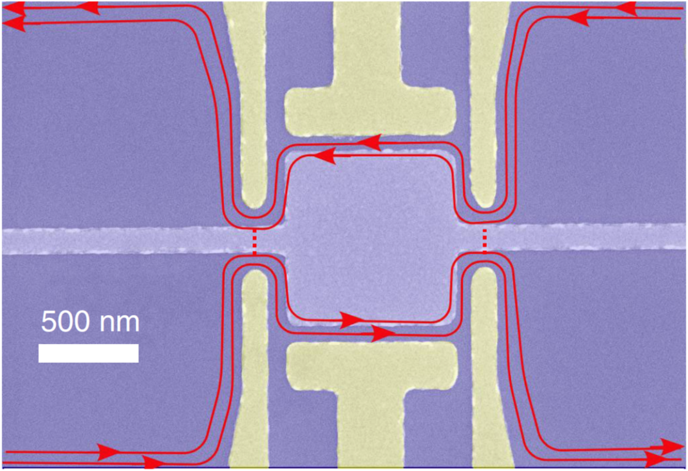
$$\phi_j(t) = \int_{\lambda(0)}^{\lambda(t/T)} A \tag{24}$$

where $A = i\langle j, \lambda(t/T)|\nabla_\lambda|j, \lambda(t/T)\rangle$ and is called the Berry connection.

7. What you need to know about the quasi-holes

- Region of charge depletion $\rightarrow$ has an "effective" $+$e/2s$+$1 charge
- Is a "defect" or "vortex point" $\rightarrow$ Obstruction to gauging, i.e Berry connection is not well defined at hole-coordinate
- Has **fractional** statistics $\rightarrow$ adiabatic transport around the hole-coordinate leads to fractional phase (mod 2π)

The quasiholes must be **anyons**![7]



[11]

[11] J. Nakamura, S. Liang, G.C. Gardner, and M.J. Manfra, https://doi.org/10.1103/PhysRevX.13.041012

There are contact points(where the edges come close together) where the edge modes can "hop". This introduces an tunneling current in the total current,

$$I = \frac{\nu e^2}{hV} + |I_t| e^{i\theta} \tag{25}$$

From this the phase accumulated by edge excitations is given by,

$$\frac{\theta}{2\pi} = \frac{e^*}{e} \frac{A_I B}{\Phi_0} + N_{qp} \frac{2\theta_a}{2\pi} \tag{26}$$

which tells us the **number of bulk anyons** in the system!

Anyon transport in the Plasma analogy

Starting from the quasi-hole excited state we can map to the plasma picture:

$$\Psi_{excited}[\eta] \leftrightarrow \mathcal{Z}_{excited} = \int \prod_{h} dz_i \, e^{-\frac{1}{m} U([z_i],\eta)} \tag{27}$$

where $U([z_i], \eta) = -2m^2 \sum_{i<j} \ln |z_i - z_j| - 2m \sum_i \ln |z_i - \eta| + \frac{m}{2} \sum_i \frac{|z_i|^2}{4l_0^2}$

Our mean density is then given in the form,

$$\bar{\rho}[\eta] = \frac{\langle e^{-\beta U([z_i],\eta)} \rangle}{\mathcal{Z}_{excited}} \tag{28}$$

Thank you for listening

References

[1] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West. "Compressibility of the two-dimensional electron gas: Measurements of the zero-field exchange energy and fractional quantum Hall gap". In: *Phys. Rev. B* 50 (3 July 1994), pp. 1760–1778. DOI: 10.1103/PhysRevB.50.1760. URL: https://link.aps.org/doi/10.1103/PhysRevB.50.1760.

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[4] S. M. Girvin, A. H. MacDonald, and P. M. Platzman. "Magneto-roton theory of collective excitations in the fractional quantum Hall effect". In: *Physical Review B* 33.4 (1986). Received 16 September 1985, pp. 2481–2494. DOI: 10.1103/PhysRevB.33.2481.

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[8] Noah L. Samuelson et al. "Anyonic statistics and slow quasiparticle dynamics in a graphene fractional quantum Hall interferometer". In: *To be completed* (2024). Preliminary or unpublished work if applicable.

[9] D. J. Thouless et al. "Quantized Hall Conductance in a Two-Dimensional Periodic Potential". In: *Phys. Rev. Lett.* 49 (6 Aug. 1982), pp. 405–408. DOI: 10.1103/PhysRevLett.49.405. URL: https://link.aps.org/doi/10.1103/PhysRevLett.49.405.

[10] X. G. Wen. "Chiral Luttinger liquid and the edge excitations in the fractional quantum Hall states". In: *Phys. Rev. B* 41 (18 June 1990), pp. 12838–12844. DOI: 10.1103/PhysRevB.41.12838. URL: https://link.aps.org/doi/10.1103/PhysRevB.41.12838.

Commutation Relations

We can canonically quantize our symmetric gauge Hamiltonian in the following way,

$$\mathcal{H}_{Landau} = \frac{\hbar^2 \eta^2}{2 m_e l_0^4} \tag{29}$$

where

$$\eta = \frac{1}{2}\mathbf{r} + \frac{l_0^2}{\hbar}\hat{\mathbf{z}} \times \mathbf{p} \tag{30}$$

$$[\eta_x, \eta_y] = i l_0^2 \tag{31}$$

Now promoting η ("cyclotron coordinate") to an operator where $\eta = \frac{\eta_x + i\eta_y}{\sqrt{2} l_0}$ and obeying,

$$\left[\eta, \eta^\dagger\right] = 1 \tag{32}$$

we get,

$$\mathcal{H}_{Landau} = \left(\eta^\dagger \eta + \frac{1}{2}\right) \frac{eB\hbar}{m_e} \tag{33}$$

Remember that in 2+1D our Landau levels are degenerate. The degeneracy is due to another canonical pair, $\mathbf{R}$, the "guiding center coordinate" where,

$$\mathbf{R} = \frac{1}{2}\mathbf{r} - \frac{l_0^2}{\hbar}\hat{z} \times \mathbf{p} \tag{34}$$

$$[R_x, R_y] = -il_0^2 \tag{35}$$

$$[\eta, \mathbf{R}] = 0 \tag{36}$$

$$[\mathcal{H}_{Landau}, \mathbf{R}] = 0 \tag{37}$$

Lastly $\mathbf{r} = \eta + \mathbf{R}$

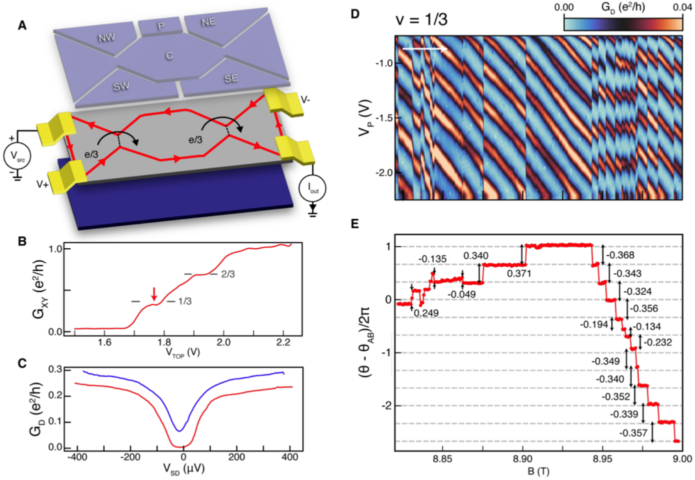

Figure: Young et al, arXiv preprint arXiv:2403.19628[12]

[12] Noah L. Samuelson et al. "Anyonic statistics and slow quasiparticle dynamics in a graphene fractional quantum Hall interferometer". In: *To be completed* (2024). Preliminary or unpublished work if applicable

Integer Quantum Hall Effect

1. Kubo formula:

$$\sigma_{xy} = -\frac{e^2}{\hbar} \sum_{\text{occ. bands}} \int_S \frac{d^2\theta}{(2\pi)^2} \, \text{Im} \left\langle \frac{\partial \psi_{n\theta}}{\partial \theta_x} \middle| \frac{\partial \psi_{n\theta}}{\partial \theta_y} \right\rangle \tag{38}$$

2. A "topological invariant":

$$C = \sum_{\text{occ. bands}} \int_S \frac{d^2\theta}{(2\pi)^2} \, \text{Im} \left\langle \frac{\partial \psi_{n\theta}}{\partial \theta_x} \middle| \frac{\partial \psi_{n\theta}}{\partial \theta_y} \right\rangle \tag{39}$$

C is referred to as the "Chern number", "winding number" or TKNN[13] invariant and $C \in \mathbb{Z}$.

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[13] D. J. Thouless et al. "Quantized Hall Conductance in a Two-Dimensional Periodic Potential". In: *Phys. Rev. Lett.* 49 (6 Aug. 1982), pp. 405–408. DOI: 10.1103/PhysRevLett.49.405. URL: https://link.aps.org/doi/10.1103/PhysRevLett.49.405