A Brief Introduction to Group Theory

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What Every Physicist Should Know

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Part I

Definitions and Examples

What even is a group???

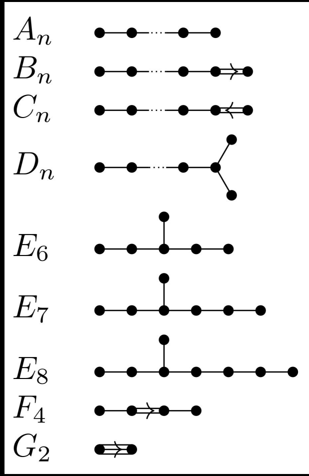
I’ll tell you what Group Theory is, in a word:
IT’S CONFUSING!!!!

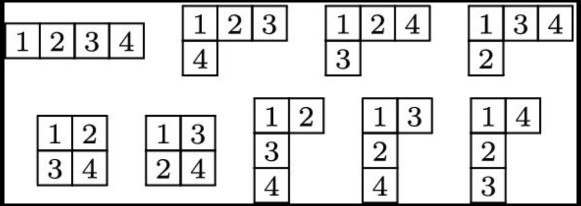
All my life I’ve heard “Group Theory this... Group theory that....” WHAT ARE THEY EVEN TALKING ABOUT???!!!

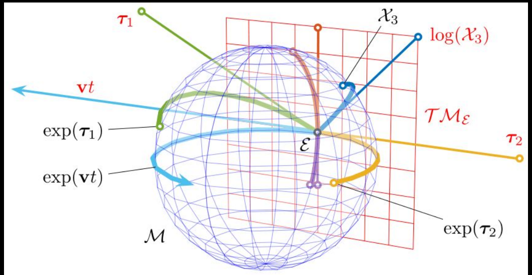
“At some point, math and physics become so abstract and detached from any real world application that they, in essence, become akin to numerology and alchemy, encoded in runes, hieroglyphics, and spell diagrams.”
- Myself (only half joking)

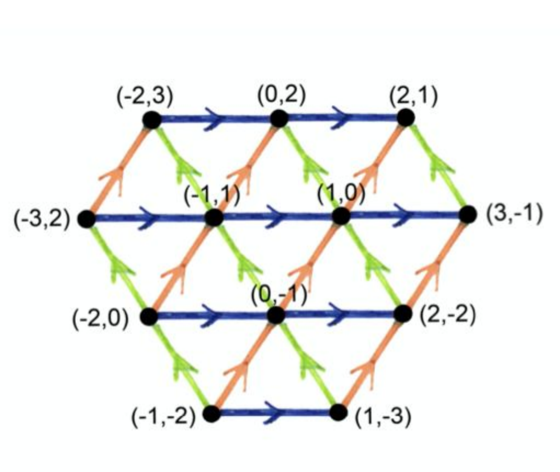
What even is a group???

THIS IS REAL GROUP THEORY DONE BY REAL GROUP THEORISTS AND THEY EXPECT US TO TAKE THEM SERIOUSLY?!



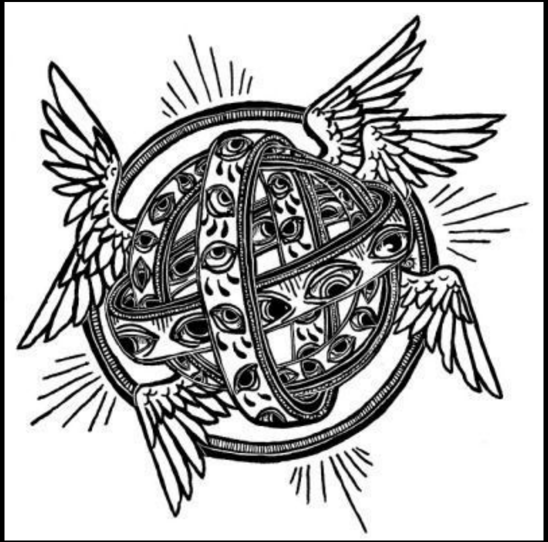






What even is a group???

Be not afraid...



Groups

Definition

A group $<G, *>$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

1) Associativity of $*$:

   For all a, b, $c \in G$, we have $(a * b) * c = a * (b * c)$.

2) Identity element e for $*$:

   There is an element e in G such that for all $x \in G$, $e * x = x * e = x$

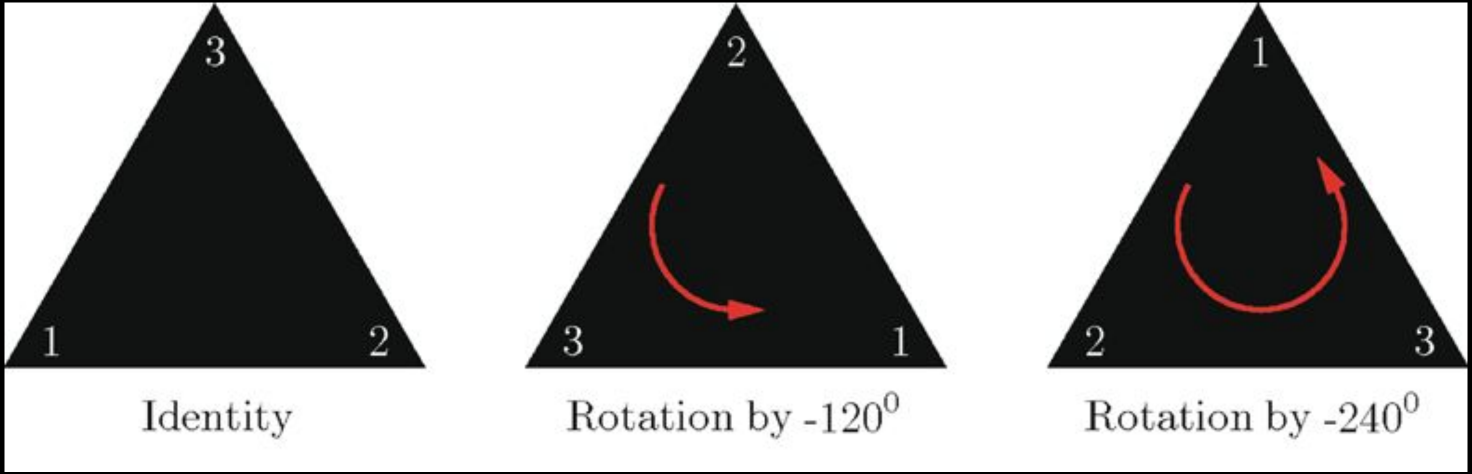
3) Inverse a' of a:

   For each $a \in G$, there is an element a' in G such that

   $$a * a' = a' * a = e.$$

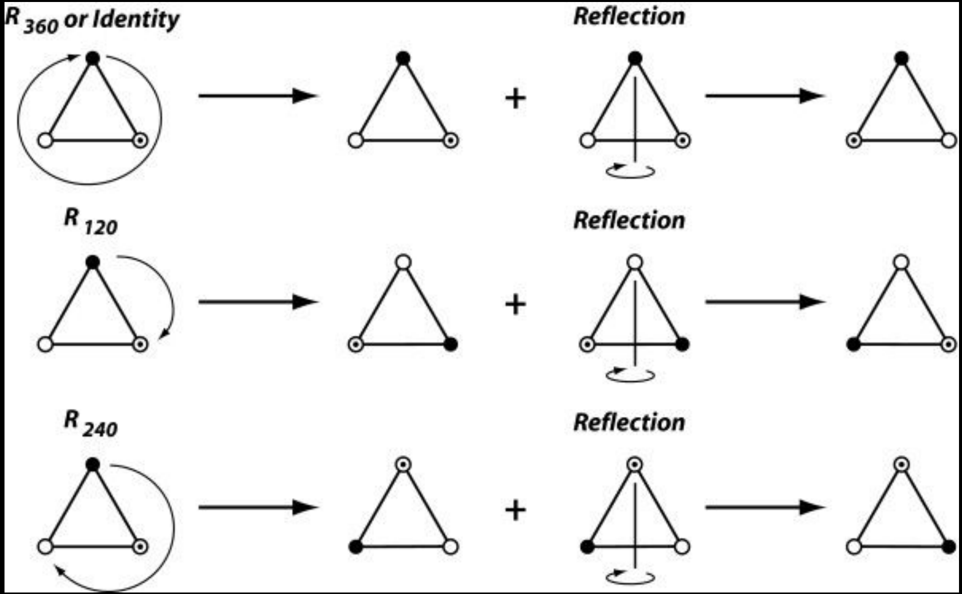
Example 1: Triangle Symmetries

. The set of all rotations on a triangle that leave it invariant with composition as the operation



Example 1: Triangle Symmetries

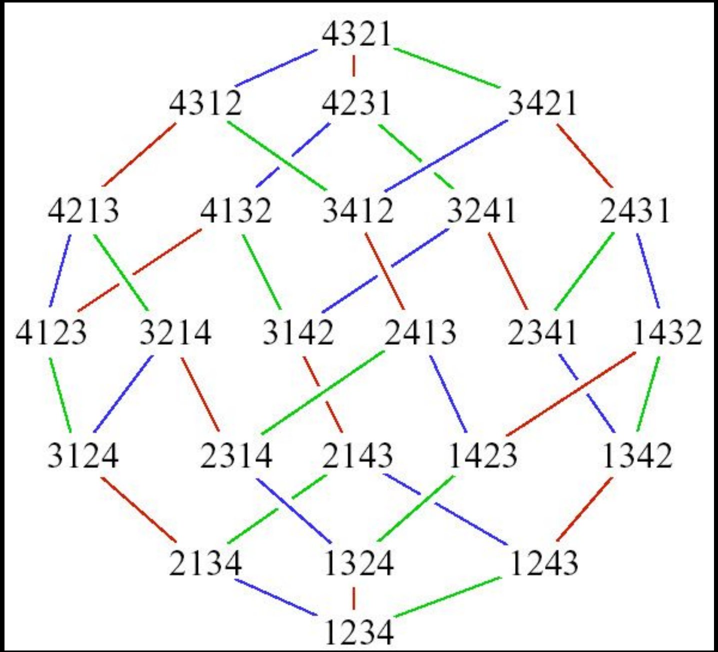
. Include reflections and we have found all symmetries
. Note that rotations are a **subgroup** of this larger group



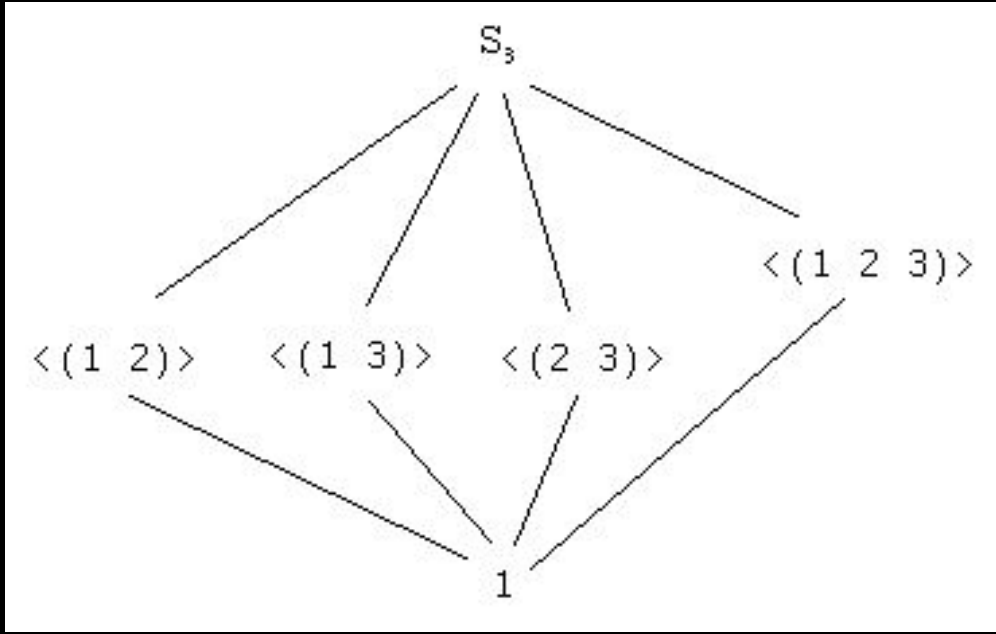
Example 2: Number Combinations

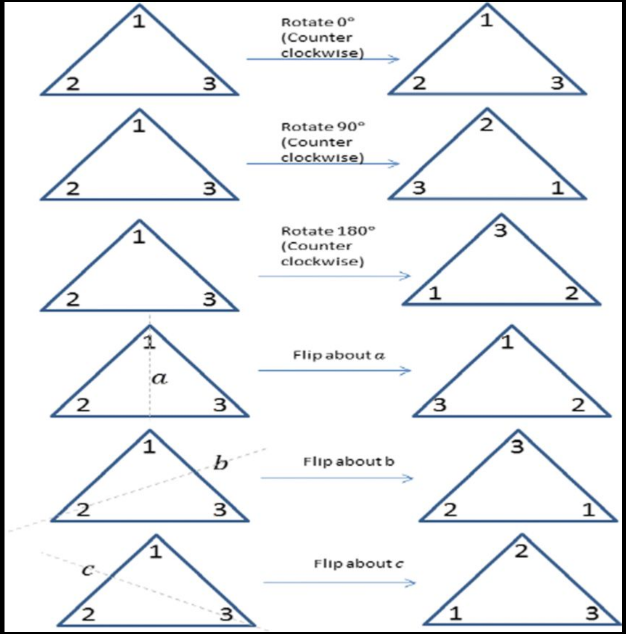
. The set of all actions which change the order of numbers in the set {1234} is a group with composition as the operation

. Such groups are called S_n, the **order** is always n!



S_n and Shape Symmetry Correspondence





Example 3: SO(N)

3. SO(N) -
   > SO(N) is the "**Special Orthogonal Group**"
   > SO(N) is a subgroup of orthogonal matrices
   > this is the group of rotational symmetries in N dimensions
   > so SO(3) is the group of all spherical rotational symmetries

$$SO(N) = \{ A \in O(N) \mid \det(A) = 1 \}$$

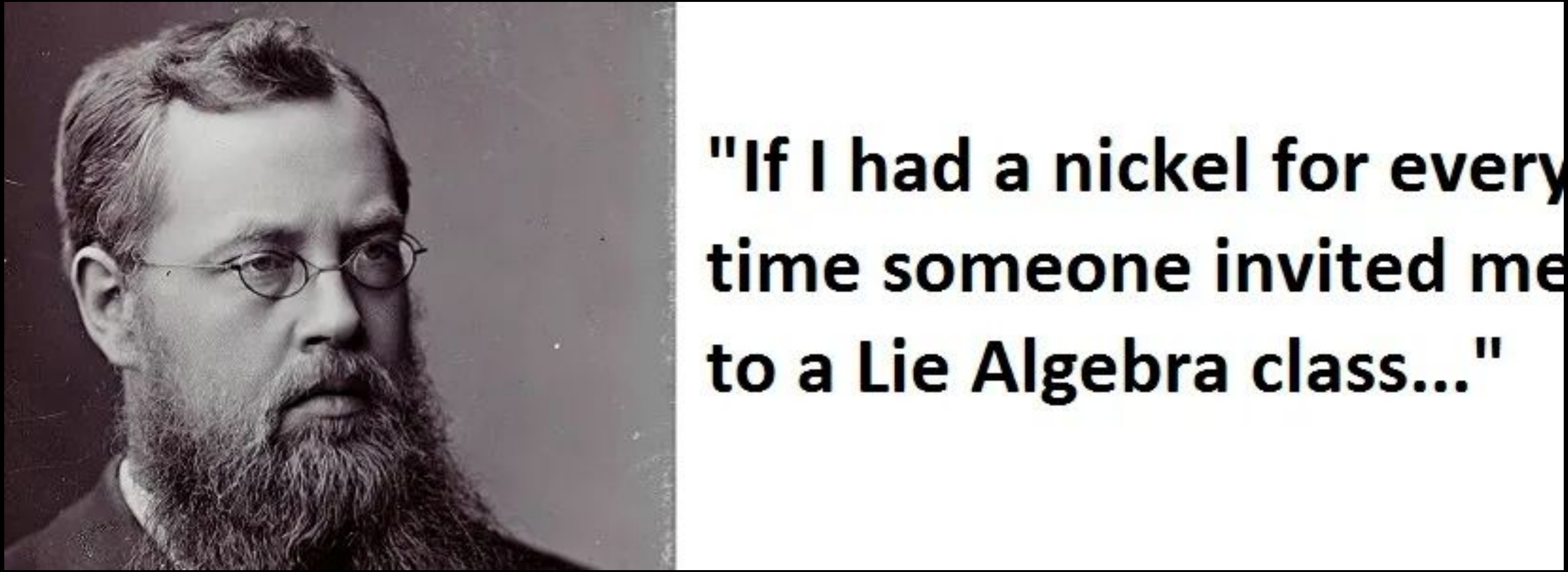
Representations

. Every group is equivalent to some set of matrices
. This is called a **representation**
. For example, we saw that S_3 is equivalent to 120 degree rotational matrices and the reflection matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Lie Algebras



Lie Algebras

$$\mathfrak{g} = \{ A \in M_{n\times n}(\mathbb{F}) \mid \exp(A) \in G \}$$

$$\det(e^{A}) = e^{\operatorname{tr}(A)}$$

Special Linear Group: $SL(n) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}(n) = \{A \in \mathfrak{gl}(n) : \operatorname{tr} A = 0\}$
Special Ortogonal Group: $SO(n) = \{A \in SL(n, \mathbb{R}) : AA^T = I\}$	$\mathfrak{so}(n) = \{A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0\}$
Special Unitary Group: $SU(n) = \{A \in SL(n, \mathbb{C}) : A\overline{A}^T = I\}$	$\mathfrak{su}(n) = \{A \in \mathfrak{sl}(n, \mathbb{C}) : A + \overline{A}^T = 0\}$

Lie Algebras

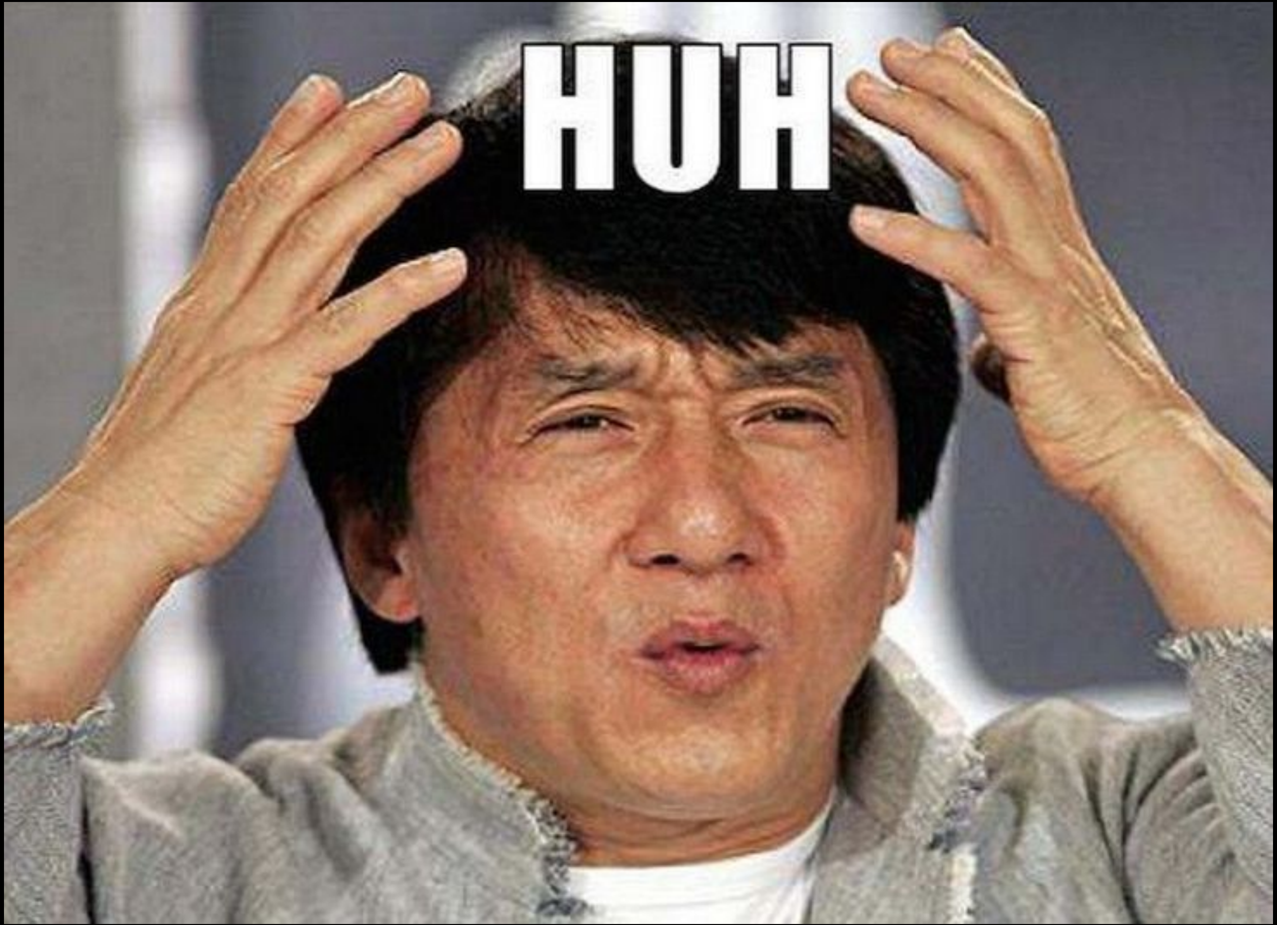
. The **Generators** of a Lie Algebra are basis vectors which satisfy certain commutation relations with the other basis vectors

. These commutation relations define the Lie Algebra and the commutators are related by **structure constants**

. a Lie Algebra representation preserves this structure

Let $\mathrm{span}(\{v_1, v_2, \dots, v_n\}) = \mathfrak{g}$

$[v_i, v_j] = f_{ijk} v_k$;


HUH

Reflection time...

Questions?

Part II

Applications in Pure Mathematics

(OMITTED)

Part III

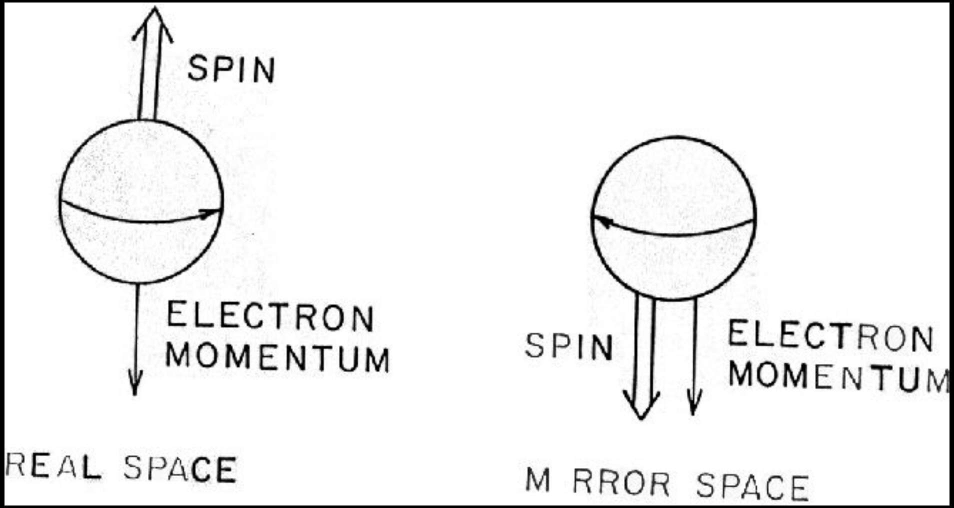
Applications in Physics

Get ready cuz we’re taking off!!!!

Basic Principles of Symmetry

. Symmetry motivates many of our favorite physics results and is a guiding principle in many situations

> Gauge symmetry
> Noether's Theorem
> Bloch's Theorem
> Wigner-Eckart Theorem
> Parity symmetry



so(3) and Pauli Matrices

Let $\text{span}(\{J_1, J_2, J_3\}) = \mathfrak{so}(3)$

$[J_1, J_2] = J_3$, $[J_2, J_3] = J_1$

$[J_3, J_1] = J_2$; $[J_i, J_j] = \varepsilon_{ijk} J_k$

Note: $J_i \to -\mathbb{i} J_i \Rightarrow [-\mathbb{i} J_i, -\mathbb{i} J_j] = -\mathbb{i}\varepsilon_{ijk} J_k$

so(3) and Pauli Matrices

$$\vec{L} = \vec{r} \times \vec{p} \Longleftrightarrow L_i = \varepsilon_{ijk} r_j p_k$$

$$[r_i, p_j] = i\hbar \delta_{ij}$$

$$\Rightarrow [L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$

so(3) and Pauli Matrices

. it turns out that i times half the Pauli spin matrices satisfy the exact same commutation relations as angular momentum

. The Stern-Gerlach experiment showed us at that magnetic moments are quantized into 2 states

. Thus, the 2D representation of angular momentum are the Pauli spin matrices!

$$\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k \Rightarrow [i\sigma_i, i\sigma_j] = -2 \varepsilon_{ijk} i \sigma_k$$

... *et violà!*

so(3) and Pauli Matrices

$$R^\dagger \hat{H} R = \hat{H}, \quad R \in SO(3)$$

$$\Rightarrow \quad R \sim \exp(\theta\, \hat{n} \cdot \vec{J}) = \exp\left(\frac{i}{2}\theta\, \hat{n} \cdot \vec{\sigma}\right)$$

$$= \exp(i\theta\, \vec{L} \cdot \hat{n})$$

note: $[\hat{H}, R] = 0 \Leftrightarrow [\hat{H}, L] = 0$

Higher Spin

. If we turn so(3) into a complex vector space, then we can construct a ladder operator algebra

. I heard that this was first done by Dirac... tell me if I am wrong

. For a spin 1/2 representation, these are given by Pauli spin matrices

$$[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$$

Note: $h=\sigma_z;\ e,f=\frac{1}{2}(\sigma_x \pm i\sigma_y);$

Higher Spin

. Let us place no restriction on the dimension of the rep

. Then in the state space of eigenvectors of the z angular momentum, the ladder operators raise are eigenvectors with raised and lowered eigenvalues

. This state space is finite dimensional

note: $H E \vec{v}_\lambda = ([H,E] + 2H)\vec{v}_\lambda = (\lambda+2) E \vec{v}_\lambda$

similarly: $H F \vec{v}_\lambda = (\lambda - 2)\vec{v}_\lambda$.

Let $\lambda = 2s$ and $2j = \max(\lambda) \Rightarrow \dim(\rho) = 2s+1$

Higher Spin

. THUS higher eigenvalues correspond to higher dimension representations of the angular momentum operators

. The higher spins go from the top value down in integer steps to the negative value

$$2D: \quad \tfrac{1}{2}H = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$3D: \quad \tfrac{1}{2}H = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \quad E = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}, \quad F = \begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}$$

Angular Momentum Addition

. Tensor product spaces are common, e.g. |1/2> x |1/2> thus we are motivated to investigate tensor product reps

. These tensor products multiply the dimension of the representation and thusly also the state space

. The dimension “breaks up” according to representation theory, e.g. 4=3+1, 9=5+3+1

. This is angular momentum addition

$$\Rightarrow \rho_{1/2} \otimes \rho_{1/2} = \rho_1 \oplus \rho_0 .$$

Hydrogen Atom

. The hydrogen atom Hamiltonian displays quirky symmetry; it has a conserved quantity so unintuitive it has been independently derived many times

. This symmetry has been known since classical mechanics for any 1/r potential

$$H = \frac{1}{2m}\vec{p}^{\,2} - \frac{Ze^2}{|\vec{r}|} \Rightarrow [\vec{L}, H] = 0,$$

$$\vec{M} = \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{Ze^2}{|\vec{r}|}\cdot\vec{r} \Rightarrow [\vec{M}, H] = 0$$

Hydrogen Atom

. In Hilbert Space, we turn observables into operators

$$\Rightarrow [N_i, L_j] = i\hbar\varepsilon_{ijk}N_k, \quad [N_i, N_j] = i\varepsilon_{ijk}L_k$$

Change basis: $\vec{I}, \vec{K} = \frac{1}{2}(\vec{L} \pm \vec{N})$

$$\Rightarrow [I_i, K_j] = 0, \quad [I_i, I_j] = i\hbar\varepsilon_{ijk}I_k,$$

$$[K_i, K_j] = i\hbar\varepsilon_{ijk}K_k$$

Hydrogen Atom

- The symmetry gives rise to chemistry

$$I^2 - K^2 = \frac{1}{2}\vec{L}\cdot\vec{N} = 0 \Rightarrow s_I = s_K$$

Denote $\rho_{s_I} \otimes \rho_{s_K}$ as $(s_I, s_K) = (k, k)$, $k \in \frac{1}{2}\mathbb{Z}$

$k = 0$: $(0,0)$, "1s" $\to 0$ (1D trivial rep)

$k = 1$: $(\frac{1}{2}, \frac{1}{2})$, "2s, 2p" $\to 0 \oplus 1$

$k = 2$: $(1, 1)$, "3s, 3p, 3d" $\to 0 \oplus 1 \oplus 2$

What even is a group???

WHAT??? THIS IS GETTING TOO WEIRD TOO FAST!!!!!

Grand Unified Theories

ADVISORY NOTICE: I know nothing about this but figured I should note it

. There could be some symmetry group which everything must obey
. Some say that this is SO(10) symmetry
. String theory is endowed with many extra dimensions which call for more and more symmetries
. Imposing **supersymmetry** on particle interactions leads to some interesting results warranting a whole field of study

This concludes my talk.

Thanks for listening :3

RECAP

. Groups obey 3 axioms and consequently any symmetry action is endowed with one

. Group theory elucidates the appearances of many strange physical phenomena:

> the appearance of Pauli matrices in rotation operators and consequential angular momentum addition

> the emergence of chemistry

Resources

1. **“Introduction to Modern Algebra”** by Saul Stahl

   > builds up to Galois Theory

2. **“Group Theory in a Nutshell For Physicists”** by A. Zee

   > very thorough; full of fun stories

3. **“Lie Algebras in Particle Physics”** by H. Georgi

   > standard textbook in the subject

ASK ME ABOUT MY...

Pure Math:

1. Platonic solids and nature
2. Construction of polygons
3. Solution to quintic equation
4. Existence of complex numbers

Physics:

1. Shortcut for normal modes
2. **!!! sl(3,C) and Gell-Mann matrices**
3. Spinors and Dirac equation